AN EXTENDED PARTIAL GEOMETRY

By Damaraju Raghavarao

Punjab Agricultural University, Hissar

1. *Summary and Introduction.* Bose (1) has defined the partial geometries and showed that many of the well known 2 associate class association schemes are particular cases of the association scheme defined by the geometry. In this paper we extend the concept of partial geometry and obtain a new 3-associate class association scheme. Cubic association scheme defined by Raghavarao and Chandrasekhararao (4) will be a particular case of the association scheme we obtain by means of our extended partial geometry.

2. *Definition of a generalized partial geometry.* Consider a system of v undefined points and b undefined lines satisfying the axioms.

A_1 Any two points are incident with not more than one line.

A_2 Each point is incident with r lines.

A_3 Each line is incident with k polnts.

A_4 Given a point P, the lines not passing through P can be divided into 2 disjoint sets S_1 and S_2 with cardinals μ_1 and μ_2 such that every line of the set S_1, can be intersected by exactly one line passing through P and no line of set S_2 can be intersected by a line passing through P.

Let P be any point and let us number the r lines passing through P as $1, 2, \ldots, r$. Number the points lying on each of these lines (excluding the point P) in an arbitrary way from $1, 2, \ldots, k-1$. Now the points on these r lines (excluding the point P) can be denoted by (i, j), where i stands for the number of the line through P and hence runs from i to r and j is the number of the point on the ith line from P and hence runs from 1 to $k-1$. Through each point (i, j) there pass $r-1$ lines other than the ith line through P and we call these $r-1$ lines as the pencil $\{i, j\}$. The lines of the pencil $\{i, j\}$

can be arbitrarily numbered from 1, 2, . . . , r-1 and the tth line of the pencil $\{i, j\}$ can be denoted as $\{i, j\,;\, t\}$. No line of the pencil $\{i, j\}$ meets a line of pencil $\{i, j'\}$ where $j \neq j'$. In fact, if $\{i, j\,;\, t\}$ and $\{i, j'\,;\, t'\}$ where $j \neq j'$ intersect in Q, then two lines passing through Q intersect the ith line through P and A_4 is violated. Lines of the pencils $\{i, j\}$ and $\{i', j'\}$ where $i \neq i'$ may intersect and here we stipulate.

A_5 Exactly one line of pencil $\{i, j\}$ intersects exactly one line of pencil $\{i', j'\}$ where $i \neq i'$ and further if $\{i, j\,;\, t\}$ intersects a line of $\{i', j'\}$ where $i \neq i'$, then it intersects one line from each of the pencils $\{i', j''\}$ where $i' \neq i$ and $j'' = 1, 2, \ldots, k-1$.

It is to be remarked here that A_5 is stated in terms of the pencil $\{i, j\}$ for clarity of expression only. This is not to be misunderstood that only one point P and the pencils thereby defined should satisfy A_5. With respect to every point of the system under consideration, pencils similar to $\{i, j\}$ should be defined and the lines of these pencils should satisfy A_5.

Definition 2·1. *A* system of undefined points, b undefined lines and an incidence relation satisfying axioms A_1 to A5 will be called an extended partial geometry with parameters r, k, 0 and 1 ; and is symbolically denoted by $[r, k\,;\, 0, 1]$.

As an illustration consider a system containing 14 points and 28 lines with an incidence relation as given by figure 1. We can easily verify that this system satisfies axioms A_1 to A_5 and hence is an extended partial geometry with parameters 4, 2, 0 and 1.

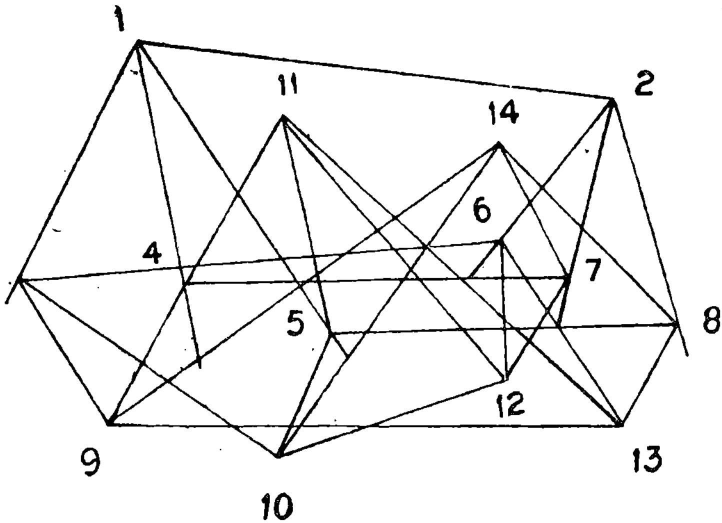
3. *Some lemmas on the structure of* $[r, k : 0, 1]$. In this section we prove some lemmas which will be useful for us to prove our main theorems in the next section.

Lemma 3·1. $[r, k\,;\, 0, 1]$ the lines do not form a triangle.

For, otherwise axiom A_4 will be violated.

Lemma 3·2. In $[r, k\,;\, 0, 1]$ if P is any point and $\{i, j\}$ are pencils of $r-1$ lines as defined in section 2, through any point Q not incident with any line through P, there pass either two or no lines of the pencils $\{i, j\}$.

This is a consequence of axiom A_5.



[4, 2 ; 0, 1]

Fig. 1.

Lemma 3·3. In [r, k ; 0, 1] the lines do not form a pentagon.

Proof. We shall prove this result by induction on r and for every k. When $r=1$, there is nothing to prove.

When $r=2$, let, if possible, the pentagon $ABCDE$ be formed.

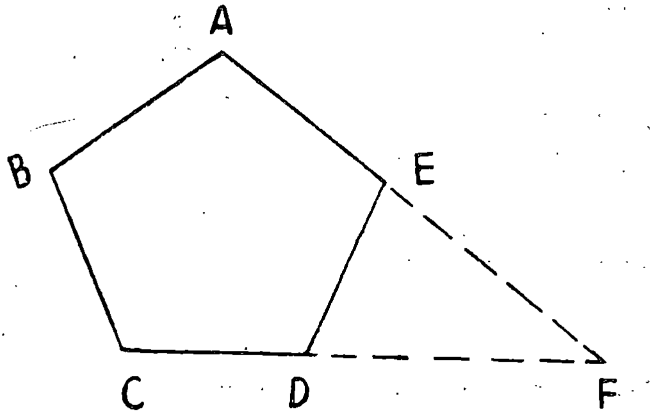


Fig. 2.

Through the point B there pass two lines BA and BC and hence from axiom A_5, one line from A other than AB must meet one line from C other than BC. Since there pass only two lines through every point, the lines AE and CD must meet in a point, say, F. Then there forms the triangle DFE in the geometry, a contradiction to Lemma 3·1. Hence there does not exist a pentagon in the geometry [2, k ; 0, 1].

Let us now assume that if there pass r lines through every point of the geometry, there does not exist a pentagon and show that if a pentagon exists when there pass $r+1$ lines through a point, there was already a pentagon when r lines only pass through a point, a contradiction to the induction hypothesis ; or one of the axioms will be violated.

Let P be any point of the geometry ; (i, j) and $\{i, j\}$ for $i=1, 2, \ldots, r$ and $j=1, 2, \ldots, k-1$ be the points and pencils of lines as defined in section 2. Let another line numbered $(r+1)$th pass through P and number the points on this line arbitrarily as $(r+1, j)$, $j=1, 2, \ldots, k-1$. To each pencil of lines $\{i, j\}$ for $i=1, 2, \ldots, r$; $j=1, 2, \ldots, k-1$ a new line will be added which will be denoted by $\{i, j ; r\}$. The lines of the pencil $\{r+1, j\}$ be denoted by $\{r+1, j ; t\}$ for $t=1, 2, \ldots, r$. If a pentagon exists when there pass $r+1$ lines through every point of the geometry, we can without loss of generality assume that it be formed by the $(r+1)$th and ith lines through P, $\{r+1, j ; t\}$, $\{i, j' ; t'\}$ and AB, where AB is the line joining A and B which are incident respectively on $\{r+1, j ; t\}$ and $\{i, j' t'\}$. We distinguish two cases and discuss them separately.

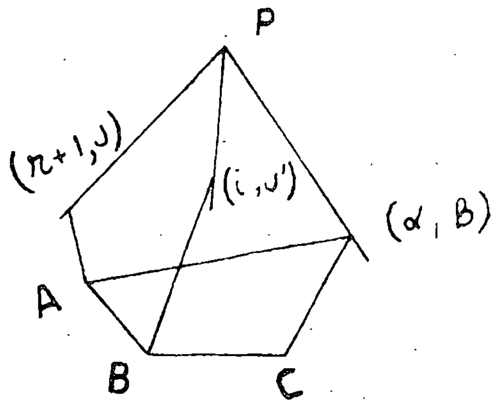


Fig. 3.

Case (i). $t' \neq r$. Clearly A must lie on some line of the form $\{\alpha, \beta ; r\}$ $\alpha=1, 2, \ldots, r$; $\beta=1, 2, \ldots, k-1$ as shown in fig. 3.

If $\alpha=i$, $\beta=j'$, then a triangle forms with the lines $\{i, j' ; t'\}$, $\{i, j' ; r\}$ and AB and if $\alpha=i$, $\beta\neq j'$, AB is a line of some pencil $\{i'', j''\}$ for $i''=1, 2, \ldots, r+1$; $j''=1, 2, \ldots, k-1$. In fact, through the point (i, j') there pass two lines ; (i) $\{i, j' ; t'\}$ and the ith line through P; and the line BA intersects $\{i, \beta\ ; r\}$ at the point A. Hence from axiom A_5, BA should intersect one line from each of the pencils $\{i, j\}$ for $j=1, 2....k-1$ and one line through the point P other than the ith line through the point P. In either case, Lemmas 3.1 or 3.2 are violated. Thus we should have $\alpha \neq i$.

Now a line B must intersect some line $\{\alpha, \beta\ ; t''\}$ $t'' \neq r$, in a point, say, C. B is an old point and AB is a new line, which implies that BC is an old line. Hence there was already a pentagon existing when there pass only r lines through every point of the geometry.

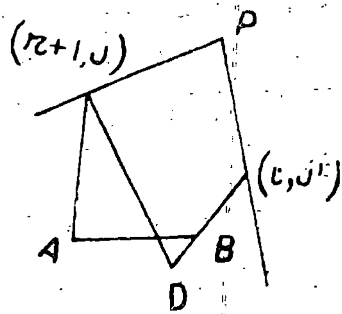


Fig. 4.

Case (*ii*) $t'=r$. Let $\{i, j'\ ; r\}$ and $\{r+1, j\ ; t''\}$ intersect in a point, say, D, as shown in figure 4. Through B, there pass two lines AB and $\{i, j'\ ; r\}$ and the line $\{r+1, j\ ; t\}$ intersects the line $\{r+1, j\ ; t''\}$ at the point $(r+1, j)$. Hence by the axiom A_5, the line $(r+1, j\ ; t\}$ should intersect a line passing through (i, j') other than $(i, j'\ ; r)$. Thus two lines of the pencil $\{i, j'\}$ intersect two lines of the pencil $(r+1, j\}$ a contradiction to the axiom A_5.

Taking these two cases into consideration, we conclude that there does not exist a pentagon when there pass $r+1$ lines through every point of the geometry. Thus our lemma is proved.

Lemma 3.4. Let P be any point of $[r, k\ ; 0, 1]$ and $\{i, j\}$, $i=1, 2, \ldots, r\ ; j=1, 2, \ldots, k-1$ be the pencils of lines as defined in section 2. If Q and R are any two points lying on some lines of some pencils $\{i, j\}$, $i=1, 2, \ldots, r\ ; j=1, 2, \ldots, k-1$; then either Q and R are not incident on a line or they lie on some line belonging to some pencil $\{i, j\}$, $i=1, 2, \ldots, r\ ; j=1, 2, \ldots, k\text{-}1$.

Proof. We distinguish 4 cases.

Case (*i*). Q and R are incident on some line $\{i, j\ ; t\}$ in which case the assertion is trivially true.

Case (*ii*). Q and R lie respectively on the lines $\{i, j\ ; t\}$ and $\{i, j\ ; t'\}$ where $t \neq t'$. If Q and R are joined by a line, then a triangle forms with vertices (i, j), Q and R, a contradiction to Lemma 3.1. Thus Q and R are not incident on a line.

Case (*iii*). Q and R lie respectively on the lines $\{i, j\ ;\ t\}$ and $\{i, j'\ ;\ t'\}$ where $j \neq j'$. From axiom A_5 it should belong to one of the pencils $\{i' \neq j\}$, $i'=1, 2, \ldots, r\ ; i,\ i' \neq i\ j=1, 2, \ldots, k-1$; if Q and R are connected.

Case (*iv*) Q and R lie respectively on the lines $\{i, j\ ;\ t\}$ and $\{i', j'. t'\}$ where $i \neq i'$. If Q and R are joined, then a pentagon forms with vertices P, (i, j), (i', j'), Q and R, a contradiction to Lemma 3.3. Thus Q and R are not incident on a line.

These four cases exhaust all possible ways of choosing Q and R and thus the lemma is proved.

1. *Main theorems.* In this section we prove the following two theorems.

Theorem 4.1. In a $[r, k\ ;\ 0, 1]$ geometry, we can define a three associate class association scheme for the points by defining the associate classes as follows : With respect to any point, P, the first associates are those points which are incident on some line passing through P ; the second associates are those points through which there pass some line intersecting some line through P ; third associates are those points which are neither first nor second associates. The parameters of the association scheme are

$$n_1=r(k-1),\ n_2=r\ (r-1)\ (k-1)^2/2,$$

$$n_3=(r-1)\ (r-2)\ (k-1)^3/2,$$

$$P_1=\begin{Bmatrix} k-2 & (r-1)(k-1) & 0 \\ & (r-1)(k-1)(k-2) & (r-1)(r-2)(k-1)^2/2 \\ & & (r-1)(r-2)(k-1)^2\ (k-2)/2 \end{Bmatrix}$$

$$P_2=\begin{Bmatrix} 2 & 2(k-2) & (r-2)(k-1) \\ & (k-2)^2+(r+1)(r-2) & (r+1)(r-2)(k-1)(k-2)/2 \\ & (k-1)/2 & (r-1)(r-2)(k-1)(k-2)^2/2 + (r-2)(r-3)(k-1)^2/2 \end{Bmatrix}$$

$$P_3=\begin{Bmatrix} 0 & r & r(k-2) \\ & r(r+1)(k-2)/2 & r[(r-1)(k-2)^2+(r-3(k-1)]/2 \\ & & \{(r-1)(r-2)(k-1)^3/2\}-1 - r(k-2) - r\{(r-1)(k-2)^2 + (r-3)(k-1)\}/2 \end{Bmatrix}$$

Theorem 4.2. The lines and points of $[r, k\ ;\ 0, 1]$ form a three associate class Partially Balanced Incomplete Block (PBIB) design (cf. [2] ; [3]) if they are respectively identified with blocks and treatments. The association scheme of the design is as given by Theorem 4.1 and the remaining parameters are

$$v = 1 + n_1 + n_2 + n_3 = 1 + r(k-1) + r(r-1)(k-1)^2/2$$
$$+ (r-1)(r-2)(k-1)^3/2,$$
$$b = vr/k,\ r,\ k,\ \lambda_1 = 1,\ \lambda_2 = 0 = \lambda_3.$$

Theorem 4.2 is an immediate consequence of theorem 4.1 and hence we prove theorem 4.1 now. Clearly there are $n_1 = r(k-1)$ first associates for every point. The second associates of P are those points lying on the lines $\{i, j\,;\, t\}$ for $i=1, 2, \ldots, r\,;\, j=1, 2, \ldots, k-1\,;\, t=1, 2, \ldots, r-1$. From Lemma 3.2, it follows that each second associate of P is incident on two such lines. Hence the number of second associates of P is $n_2 = r(r-1)\,(k-1)^2/2$. Through each second associate of P there pass $r-2$ lines other than the lines of the pencils $\{i, j\}$, $i=1, 2, \ldots, r\,;\, j=1, 2, \ldots, k-1$ and on each of these lines there are $k-1$ third associates of P. By noting that each of the third associates of P is incident on r such lines, we have $n_3 = r\,(r-1)(r-2)(k-1)^3/2r = (r-1)(r-2)(k-1)^3/2$, to be the number of third associates of P.

The parameters $p'_{11}, p'_{12}, p'_{22}$ can easily be verified to take the values as anunciated in the theorem. We now prove the expression for p^2_{22}. Let P and Q be the second associates and Q lie on the lines $\{i, j\,;\, t\}$ and $\{i', j'\,;\, t'\}$. Let $\{i, j\,;\, t\}$ intersect one line from each of the pencils $\{i', j_l\}$ $j_l = 1, 2, \ldots, k-1\,;\, jl \neq j'$ in the points Ql. The points that are incident on the lines joining (i, jl) and Ql excluding the points (i', jl) and Ql are the common second associates of P and Q for $jl = 1, 2, \ldots, k-1\,;\, jl \neq j'$. These points are $(k-2)^9$ in number. Similarly, the line $\{i', j'\,;\, t'\}$ intersects one line from each of the pencils $\{i, jl\}$, $jl = 1, 2, \ldots, k-1\,;\, j_l \neq j$ in the points Rl. The poinis thnt are incident on the line joining (i, jl) and Rl excluding the points (i, jl) and R are also common second associates of both P and Q for $jl = 1, 2, \ldots, k-1\,;\, jl \neq j$. But these points are just the same as enumerated earlier. The points lying on the lines $\{i, j\,;\, t''\}$, $t'' = 1, 2, \ldots, r-1$, $t'' \neq t$ are common sccond associates of both P and Q and also the points lying on the lines $\{i', j'\,;\, t''\}$, $t'' = 1, 2, \ldots, r-1$, $t'' \neq t'$. These are $2\,(r-2)\,(k-1)$ in number. Through the point Q, there pass $r-2$ lines other than the lines $\{i, j\,;\, t\}$ and $\{i', j'\,;\, t'\}$ and we call these $r-2$ lines, the pencil of lines through Q and denote them by $\{Q\}$. Let R be any point on some line of the pencil $\{Q\}$. Then on each line passing through R there can be at most one point of the lines of the pencils $\{i'', j''\}$, $i''=1, 2, \ldots, r$; $i'' \neq i$; $i'' \neq i', j''=1, 2, \ldots, k-1$ and if B is any point lying on some line of $\{i'', j''\}$ which is incident on some line passing through R, then B and a point R' are incident on a line where $R' \neq R$ and R' is also a point on some line of the pencil $\{Q\}$. By noting this and enumerating the common second associates of both P and Q, we get $(r-2)(r-3)(k-1)/2$ additional points. Thus, we see that p^2_{22} is as given in the theorem. The remaining parameters can also be similarly obtained and the theorem is established.

Considering the particular case $k=2$, we have corollary 4.1.1. The generalized partial geometry $[r, 2; 0, 1]$ defines a three associate class association scheme with parameters.

$$n_1=r,\ n_2=r(r-1)/2,\ n_3=(r-1)(r-2)/2$$

$$P_1=\begin{Bmatrix} 0 & r-1 & 0 \\ & 0 & (r-1)(r-2)/2 \\ & & 0 \end{Bmatrix}$$

$$P_2=\begin{Bmatrix} 2 & 0 & r-2 \\ & (r+1)(r-2)/2 & 0 \\ & & (r-2)(r-3)/2 \end{Bmatrix}$$

$$P_3=\begin{Bmatrix} 0 & r & 0 \\ & 0 & r(r-3)/2 \\ & & 0 \end{Bmatrix}$$

The assoctation scheme defined in Theorem 4.1 is cubic association scheme when $r=3$. When $r=2$, the 3 associate classes will reduce to 2 associate classes only and then it is an L_2 association scheme.

Lemma 4.1. v, the number of points in $[r, k; 0, 1]$ satisfies $v=0 \pmod{k}$

Proof.

$$v=1+r(k-1)+r\,\frac{(r-1)(k-1)^2}{2}+\frac{(r-1)(r-2)(k-1)^3}{2}$$

$$=1-r+r\,\frac{(r-1)}{2}-\frac{(r-1)(r-2)}{2} \pmod{k}$$

$$=0 \pmod{k}.$$

The constructions of partial geometries discussed in this paper opens a new combinaterial problem and the author hopes to communicate these results in a subsequent paper.

REFREENCES

1. **BRUCK, R.H.** (1963). Finite Notes. II. Uniqueness and Imbedding. Pacific Journal of Maths. Vol. **13**, 421-457.
2. **BOSE. R.C.** (1963). Partial geometries. Pacific Journal of Maths. Vol. **13**, 389-419.
3. **BOSE, R.C. and NAIR, K.R.** (1938-40). Partially balanced incomplete block designs. Sankhya, **4,** 337-372.
4. **BOSE. R.C. and SHIMAMOTO, T** (1952). Classification and analysis of partially balanced designs with two associate classes. J. Amer. Stat. Assn., **47,** 151-190.
5. **RAGHAVARAO, D. and CHANDRASEKHARA RAO, K.** (1964). Cubic designs. Ann. Math. Stat., **35**, 389-397.